Dynamics in Electrical Systems

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***Abstract*—The abstract goes here.**

***Index Terms*—differential, dynamics, electrical, equation, modeling, ordinary, system**

I. Introduction

THIS this paper is intended to sum up the research done in order to understand the Dynamics in electrical systems and their underlying differential equations.

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II. Dynamical Systems

Dynamical systems are mathematical objects used to model physical phenomena whose state (or instantaneous description) changes over time. These models are used in financial and economic forecasting, environmental modeling, medical diagnosis, industrial equipment diagnosis, and a host of other applications.

For the most part, applications fall into three broad categories: predictive (also referred to as generative), in which the objective is to predict future states of the system from observations of the past and present states of the system, diagnostic, in which the objective is to infer what possible past states of the system might have led to the present state of the system (or observations leading up to the present state), and, finally, applications in which the objective is neither to predict the future nor explain the past but rather to provide a theory for the physical phenomena. These three categories correspond roughly to the need to predict, explain, and understand physical phenomena.

A. Differential Equations

A **differential equation** is any equation which contains derivatives, either ordinary derivatives or partial derivatives. Almost every physical situation that occurs in nature can be *described* with an appropriate differential equation.

The process of describing a physical situation with a differential equation is called **modeling**.

Differential equations are generally concerned about three questions:

1) Given a differential equation will a solution exist?
2) If a differential equation does have a solution how many solutions are there?
3) If a differential equation does have a solution can we find it?

There are two types of differential equations. *Ordinary differential equations* (PDE) and *Partial differential equations*. Our study won't go into further detail about PDE and will stay focused mainly on ODE.

B. Direction Field

Understanding **direction fields** (or **slope fields)** and what they tell us about a differential equation and its solution is important and can be introduced without any knowledge of how to solve a differential equation and so can be done before the getting to actually solving them.

The direction fields are important because they can provide a *sketch of solution*, if exist, and a *long term behavior* - most of the time we are interested in general picture about what is happening, as the time passes.

III. Periodic Orbits

A periodic orbit corresponds to a special type of solution for a dynamical system, namely one which repeats itself in time. A dynamical system exhibiting a stable periodic orbit is often called an *oscillator*.

A. Limit Cycle

A **limit cycle** is an isolated closed trajectory. *Isolated* means that neighboring trajectories are not closed - they spiral either towards or away from the limit cycle. The particle on the limit cycle, appears after one period on the exact same spot. Limit cycle appears on on a plane, opposed to a periodic orbit, that happens to be a vector.

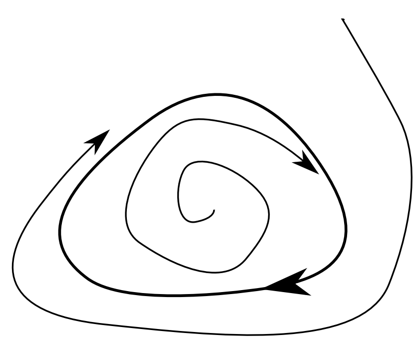

Fig. 1. Stable limit cycle. Trajectories spiral towards it.

If all neighboring trajectories approach the limit cycle, we say the limit cycle is **stable** or *attracting*, as shown on fig. 6. Otherwise the limit cycle is **unstable**, or in exceptional cases, **half-stable**. Stable limit cycles are very important scientifically as they model systems that exhibit self-sustained oscillations. In other words, these systems oscillate even in the absence of external periodic forcing.

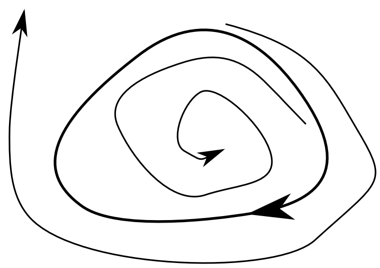

Fig. 2. Unstable limit cycle. Trajectories spiral away from it.

Of the countless examples that could be given, we mention only a few: the beating of a heart; the periodic ring of a pace maker neuron; daily rhythms in human body temperature and hormone secretion; chemical reactions that oscillate spontaneously; and dangerous self-excited vibrations in bridges and airplane wings. In each case, there is a standard oscillation of some preferred period, waveform, and amplitude. Oscillations are important part of electronics [2], too.

If the system is perturbed slightly, it always returns to the standard cycle. Limit cycles are inherently nonlinear phenomena; they cant occur in linear systems [7].

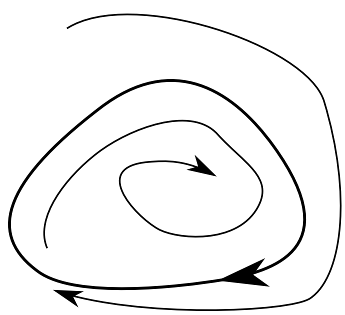

Fig. 3. Half-stable (or semi-stable) limit cycle. Attract trajectories from one side and repel them from other side.

B. Damping

Mentioning damping is important mainly because, in a real world, oscillations eventually stop, due to Newton's law of Thermodynamics (the frictional force). In electronics, there is no ideal oscillator, too - small amount of energy is lost every cycle, due to electric resistance.

Generally, the damping is linear either linear or nonlinear. As a rule of thumb, the linear one is easily modeled mathematically, obeying known rules, while the nonlinear one is not [1]. There are some use cases, where nonlinear damping is advantageous, but the research is still ongoing about this topic.

IV. Conducted Studies

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A. Liénard Equation

A nonlinear second-order ordinary differential equation

$$y'' + f(x)x' + x = 0 \quad (1)$$

This equation describes the dynamics of a system with one degree of freedom in the presence of a linear restoring force and nonlinear damping. The function f has properties

$$f(x) < 0 \quad for\, small\, |x|$$
$$f(x) > 0 \quad for\, large\, |x|$$

that is, if for small amplitudes the system absorbs energy and for large amplitudes dissipation occurs, then in the system one can expect self-exciting oscillations.

Liénard equation was intensely studied as it can be used to model oscillating circuits. Under certain additional assumptions Liénard's theorem guarantees the uniqueness and existence of a limit cycle for such a system.

B. Van der Pol Equation

One of the most well-known oscillator model in dynamics is **Van der Pol oscillator**, which is a special case of Liénard's equation (3) and is described by a differential equation

$$y'' - \mu\left(1 - y^2\right) y' + y = 0 \quad (2)$$

where y is the dynamical variable and $\mu > 0$ is a parameter. If $\mu = 0$, then the equation reduces to the equation of simple harmonic motion

$$y'' + y = 0$$

The μ parameter determines the shape of the limit cycle. As it approaches 0, it gets the shape of a circle. On the other hand, increasing the paramter, involves sharpening of the curves.

The Van der Pol equation (2) arises in the study of circuits containing vacuum tubes (triode) and is derived from earlier, Rayleigh equation [4], known also as Rayleigh-Plesset equation - an ordinary differential equation explaining the dynamics of a spherical bubble in an infinite body of liquid.

Van der Pol oscillator is **self-sustainable**, **relaxation** oscillator. Self-sustainability in this context means, that the energy is fed into small oscillations and removed from large oscillations. Relaxation means, that the energy is gradually accumulating over time and then quickly released (relaxed). In electronics jargon, the relaxation oscillator is also called a *free-running* oscillator. As already explained, it does not require neither one (monostable), nor two (bistable) inputs for transitioning between states, it ”runs” by itself, thus free-running.

C. Periodicity in Van der Pol's Oscillator

Liénard's theorem can be used to prove that the system described by Van der Pol equation (2) has a limit cycle [5]. If we want to visualize it, the one-dimensional form of equation must be first *transformed* to the two-dimensional form. Applying the Liénard transformation

$$y = x - \frac{x^3}{3} - \frac{\dot{x}}{\mu}$$

where dot indicates the time derivative, the system can be written in it's two-dimensional form [3]:

$$\dot{x} = \mu\left(x - \frac{1}{3}x^3 - y\right)$$
$$\dot{y} = \frac{1}{\mu}x$$

However, this form is not well-known. Far common form uses the transformation $y = \dot{x}$, that yields

$$\dot{x} = y$$
$$\dot{y} = \mu\left(1 - x^2\right)y - x$$

which can be plotted onto direction field, as shown on fig. 4. It is possible to see the stable limit cycle as well as trajectories from both sides attracted towards it.

The Van der Pol oscillator can be forced too, however, this work does not aim to investigate further in this direction.

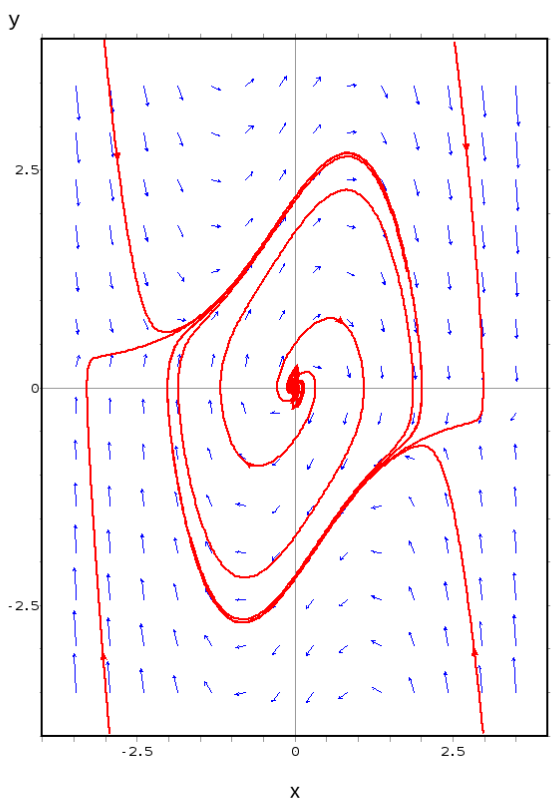

Fig. 4. Phase portrait of the unforced Van der Pol oscillator, showing a limit cycle and the direction field Parameter $\mu = 1$. The wxMaxima computing software was used for this purpose.

V. Applications in Electronics

VI. RLC Circuits

In RLC circuit exist elements like resistor; inductor and capacitor are connected across a voltage supply. All these elements are linear and passive in nature; i.e. they consume energy rather than producing it and these elements have a linear relationship between voltage and current. The RLC circuit exhibits the property of resonance in same way as LC circuit exhibits, but in this circuit the oscillation dies out quickly as compared to LC circuit due to the presence of resistor in the circuit. There are number of ways of connecting these elements across voltage supply, but the most common method is to connect these elements either in series or in parallel.

A. *Example 1.*

Consider the circuit shown in Figure 1 below, consisting of a resistor, a capacitor, and an inductor (this type of circuit is commonly called an RLC circuit). The circuit contains two energy storage elements: and inductor and a capacitor. The energy storage elements are independent, since there is no way to combine them to form a single equivalent energy storage element. Thus, we expect the governing equation for the circuit to be a second order differential equation. We will develop equations governing both the capacitor voltage, vC(t) and the inductor current, iL(t) as indicated in Figure 1. In order to

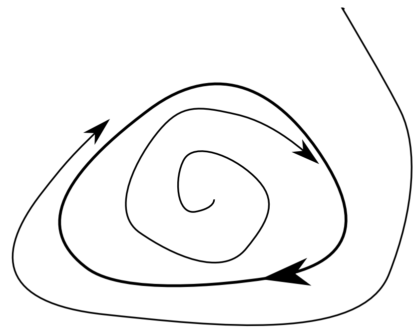

Fig. 5. Figure 1. Series RLC circuit

determine the governing equations for vC(t) and iL(t) we will attempt to write two firstorder differential equations for the system and then combine these equations to obtain the desired second order differential equation. To facilitate this process, the circuit of Figure 1 is repeated in Figure 2 with the node and loop we will use labeled. Note that we also label the current through the capacitor in terms of the capacitor voltage and the voltage across the inductor in terms of the inductor current. The voltage-current relationships for inductors and capacitors

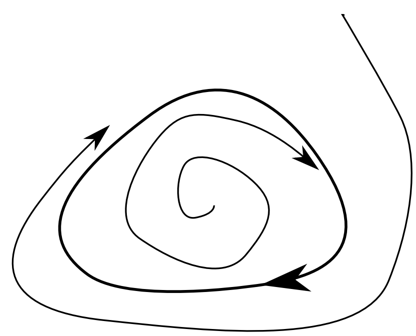

Fig. 6. Figure 2. Series RLC circuit with node and loop defined.

indicate that, in Figure 2, $i_c(t) = C\frac{dv_c(t)}{dt}$ and $v_l(t) = L\frac{di_L(t)}{dt}$. Using the latter of these relations, KVL around the indicated loop in Figure 2 provides:

$$v_s(t) = Ri_L(t) + v_c(t) + L\frac{di_L}{dt} \tag{3}$$

KCL at node A, along with the voltage-current relation for the capacitor, indicates that:

Important Tip: Equations (1) and (2) consist of two coupled first order differential equations in two unknowns: i_L(t) and v_c(t). This set of differential equations completely describes the behavior of the circuit if we are given appropriate initial conditions and the input function v_s(t) they can be solved to determine the inductor currents and capacitor voltages. Once the capacitor voltage and inductor current are known, the energy in the system is completely defined and we can determine any other desired circuit parameters. Any manipulations of equations (1) and (2) we perform subsequently in this chapter do not fundamentally increase the information we have about the circuit we will simply be trying to rearrange equations (1) and (2) to make it easier (in some ways) to interpret the circuit behavior.

We can determine the equation governing the capacitor voltage by differentiating equation (2) with respect to time to obtain an expression for the derivative of the inductor current:

Substituting equations (2) and (3) into equation (1) results in:

Rearranging this slightly results in

To determine the relationship governing the inductor current, we can again use equation (2) to write the capacitor voltage as:

where we assume that the voltage across the capacitor at time t = 0 is zero; e.g. v_c (0) = 0.

In general, we prefer not to work with a mixture of derivatives and integrals in the same equation, so we differentiate the above to obtain our final expression for i_L(t):

B. *Josephson Junctions*

Josephson junctions are superconducting devices that are capable of generating voltage oscillations of extraordinary high frequency, typically 10^{10} - 10^{11} cycles per second [8]. They consist of two superconducting layers, separated by a very thin insulator that weakly couples them, as shown on fig. 7.

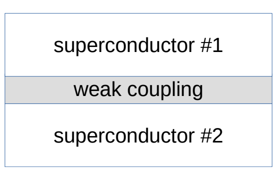

Fig. 7. The physical structure of a Josephson Junction. Shown for ilustration purposes.

Although quantum mechanics is required to explain the origin of the Josephson effect, we can nevertheless dive into dynamics of Josephson junctions in classical terms. They have been particularly useful for *experimental* studies of nonlinear dynamics, because the equation governing a single junction resembles the one of a pendulum [6].

Josephson junctions are used to detect extremely low electric potentials and are used for instance, to detect far-infrared radiation from distant galaxies. They are also formed to arrays, because there is a great potential seen in this configuration, however, all the effects are yet to be fully understood.

VII. Conclusion

The conclusion goes here.

Appendix A
Proof of the First Zonklar Equation

Appendix one text goes here.

Appendix B

Appendix two text goes here.

Acknowledgment

The authors would like to thank professor Carlos Parés for having patience with them. Another thank would go to the well-written book *Nonlinear Dynamics and Chaos, S.H. Strogatz, 2008* for introduction to and for sparking curiosity in the field of Dynamical Systems.

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